Some remarks on optimal codes

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Abstract

Abstract:

Let $A(n, d)$ (resp. $A(n, d, w)$) be the maximal cardinality of codes (resp. constant-weight codes of weight w) of length n and minimum distance d. We introduce our recent results on $A(n, d)$ and $A(n, d, w)$. We improve two values of $A(n, d)$ and twenty one values of $A(n, d, w)$.

CONTENTS

I. Coding Theory

Coding Theory

Let $B = \{0, 1\}$ be the set of binary alphabets. In coding theory, one would like to study the set $X = B^n$ of n-tuples of alphabets. We can view X from various point of views:

(a) X is a set.

(b) X is an abelian group when we give an abelian group structure on B.

(c) X is a vector space when we give a field structure on B. Sometimes we denoted this field by $\mathbb{F}_2$ and the n-dimensional vector space over $\mathbb{F}_2$ by $\mathbb{F}_2^n$.

Coding Theory-Cont.

(d) $\mathbb{F}_2^n$ is a metric space when it is equipped with the Hamming metric:

- For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{F}_2^n$, we define

$$d(x, y) = |\{i \mid x_i \neq y_i\}|.$$

Sometimes this metric space is denoted by $(\mathbb{F}_2^n, d_H)$.

(e) X is a graph where two vectors $x, y \in X$, x and y are adjacent iff $d_H(x, y) = 1$. This graph is called the Hamming graph, and it is a distance regular graph.

(f) X is an association scheme which is called the Hamming scheme.

(g) X form an affine geometry $AG(n, 2)$ of dimension n over $\mathbb{F}_2$.

What is a code?

Binary code:

- A subset $\mathcal{C}$ of B^n is called a *(binary) code* of length n.
- An element of a code $\mathcal{C}$ is called a *codeword*.

Minimum distance of a code:

- For $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ in B^n, define

$$d(X, Y) = |\{i \mid x_i \neq y_i\}|.$$

- *Minimum distance* of a code $\mathcal{C}$ is defined by

$$\min\{d(X, Y) | X, Y \in \mathcal{C}, X \neq Y\}.$$

Codes can correct errors

Three basic parameters of a code: length n, cardinality $|\mathcal{C}|$, minimum distance d. The minimum distance determine the error capability:

Theorem: A code with minimum distance d can correct up to $\left\lfloor \frac{d-1}{2} \right\rfloor$ errors.

A fundamental problem on coding theory

A natural problem:

Given n, find a code of length n having:

- large minimum distance,
- large number of codewords.

Remark: We can't get both.

A fundamental problem on coding theory-Cont.

Definition: Given n and d, define

$$\begin{aligned} A(n,d) &= \text{maximum number of codewords} \\ &\quad \text{in any code of length } n \text{ and} \\ &\quad \text{minimum distance } \geq d. \end{aligned}$$

Elementary properties of $A(n, d)$

Elementary properties of $A(n, d)$:

(1) $A(n, 2d) = A(n - 1, 2d - 1)$,

(2) $A(n, d) \leq 2A(n - 1, d)$.

These properties are useful when n or d is small.

Some upper bounds for $A(n, d)$

To investigate $A(n, d)$ for larger values of n and d, we need some theory for upper bounds on $A(n, d)$. The upper bounds which will be useful in our investigation are:

- Hamming bound
- Johnson bound
- Delsarte's linear programming bounds
- Schrijver's semi-definite programming bounds

Hamming bounds

Theorem (Hamming):

$$A(n, 2d+1) \leq \frac{2^n}{1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{d}}.$$

Quick reason: The balls of radius d centered at codewords are mutually disjoint!

Johnson bound

We need the concept of 'optimal constant-weight codes' in introducing Johnson bound:

Definition: Given n, d, and w, define

$$A(n,d,w) = \begin{array}{l}\text{maximum number of codewords}\\ \text{in any code of length n and}\\ \text{minimum distance} \geq d \text{ such that}\\ \text{each codeword has exactly w ones.}\end{array}$$

Remark: Codes in the above definition are called *constant-weight codes* of length n and weight w.

Johnson bound-Cont.

Theorem (Johnson):

$$A(n, 2d+1) \leq \frac{2^n}{1 + \binom{n}{1} + \cdots + \binom{n}{d} + \frac{\binom{n}{d+1} - \binom{2d+1}{d} A(n, 2d+2, 2d+1)}{\left[\frac{n}{d+1}\right]}}.$$

Quick reason: Hamming bound + careful consideration on spheres of radius $d+1$ centered at codewords.
Why $A(n, 2d+2, 2d+1)$ appears in this consideration?

Elementary properties of $A(n,d,w)$

Elementary properties of $A(n,d,w)$:

(1) $A(n,d,w) = A(n,d+1,w)$, if d is odd,

(2) $A(n,d,w) = A(n,d,n-w)$,

(3) $A(n,2,w) = \binom{n}{w}$,

(4) $A(n,2w,w) = \lfloor \frac{n}{w} \rfloor$,

(5) $A(n,d,w) = 1$, if $2w < d$,

(6) $A(n,d,w) \le \lfloor \frac{n}{w} A(n-1,d,w-1) \rfloor$,

(7) $A(n,d,w) \le \lfloor \frac{n}{n-w} A(n-1,d,w) \rfloor$.

Philosophy involved in Johnson's result

If we have a 'good' upper bound for $A(n, d, w)$, then it would induce a 'good' upper bound for $A(n, d)$. In this point of view, we need to develop 'coding theory' on $Y = B^{n,w}$ where $B^{n,w}$ denotes the set of binary n-vectors of weight w. We first trying to understand Y from various point of views:

(1) Y is a set.

(2) Y is equipped with a metric, since it is a subset of a metric space.

(3) Y is a graph which is called the Johnson graph.

(4) Y is an association scheme which is called the Johnson scheme.

Philosophy involved in Johnson's result-Cont.

(1) Since Y is a set equipped with a metric, Hamming type theorem can be developed for $A(n,d,w)$. In this case, the size of a ball of radius $2r$ is: $1+\binom{w}{1}\binom{n-w}{1}+\cdots+\binom{w}{r}\binom{n-w}{r}$.

(2) We can also develop Johnson type theorem for $A(n,d,w)$ by considering Hamming type theorem + careful consideration on spheres of radius t in Y.

(3) This means that we need to consider a subset of vectors in X which is at distance r from one point, and at distance t from another point.

A general definition for $A(n,d)$, $A(n,d,w)$

Definition: For a finite (possibly empty) set $\Lambda = \{(X_i, d_i)\}_{i \in I}$, where each X_i is a vector in X and each d_i is a nonnegative integer, we define

$$A(n,\Lambda,d) \quad = \quad \begin{array}{l} \text{maximum number of codewords} \\ \text{in any binary code of length } n \\ \text{and minimum distance } d \text{ such that} \\ \text{each codeword is at distance } d_i \\ \text{from } X_i \text{ for all } i \in I. \end{array}$$

A general definition-Cont.

Special case 1: $|\Lambda| = 0$ We get the usual definition of $A(n, d)$.

Special case 2: $|\Lambda| = 1$ Suppose $\Lambda = \{(X_1, d_1)\}$. By translation, we may assume that X_1 is the zero vector. Hence,

$$A(n, \Lambda, d) = A(n, d, w),$$

where $w = d_1$.

What will happen if $|\Lambda| = 2$?

Doubly-constant-weight codes

Definition:

$$T(w_1, n_1, w_2, n_2, d) \quad = \quad \begin{array}{l} \text{maximum number of codewords} \\ \text{in any code of length } n \text{ and} \\ \text{minimum distance } \geq d \text{ such that} \\ \text{each codeword has exactly } w_1 \text{ ones} \\ \text{on the first } n_1 \text{ coordinates and exactly} \\ w_2 \text{ ones on the last } n_2 \text{ coordinates.} \end{array}$$

Remark: Codes in the above definition of $T(w_1, n_1, w_2, n_2, d)$ are called *doubly-constant-weight codes.*

A general definition-Cont.

Special case 3: $|\Lambda| = 2$ Let $\Lambda = \{(X_1, d_1), (X_2, d_2)\}$. We have the following proposition.

Proposition: If $\Lambda = \{(X_1, d_1), (X_2, d_2)\}$, then

$$A(n, \Lambda, d) = T(w_1, n_1, w_2, n_2, d),$$

where $n_1 = d(X_1, X_2)$, $n_2 = n - n_1$, $w_1 = \frac{1}{2}(d_1 - d_2 + n_1)$, and $w_2 = \frac{1}{2}(d_1 + d_2 - n_1)$.

Delsarte's Linear Programming bounds

Distance distribution:

- Let $\mathcal{C}$ be a code (of length n).
- For each $i = 0, 1, \ldots, n$, define

$$B_i = \frac{1}{|\mathcal{C}|} \cdot |\{(X,Y) \in \mathcal{C}^2 | d(X,Y) = i\}|.$$

- The set $\{B_i\}_{i=0}^{n}$ is called the *distance distribution* of $\mathcal{C}$.

Important equality:

$$B_0 + B_1 + \cdots + B_n = |\mathcal{C}|.$$

Delsarte's Theorem

Theorem (Delsarte, 1973): Let $\mathcal{C}$ be a code with distance distribution $\{B_i\}_{i=0}^n$. Then

$$\sum_{i=0}^{n} P_k(n;i) B_i \geq 0$$

for each $k = 1, 2, \ldots, n$, where $P_k(n;x)$ is the *Krawtchouk polynomial* given by

$$P_k(n;x) = \sum_{j=0}^{n} (-1)^j \binom{x}{j} \binom{n-x}{k-j}.$$

Delsarte linear programming (LP) bound

Delsarte LP bound Consider B_i's as variables. Then

$$A(n,d) \leq 1 + \max\lfloor B_1 + \cdots + B_n \rfloor,$$

where the maximization is taken over all $(B_1, \ldots, B_n)$ satisfying the linear constraints of Delsarte's theorem and satisfying $B_i \geq 0$ for $i = 1, \ldots, n$.

Schrijver SDP bound

Definition (Triple distance distribution):

- Let $\mathcal{C}$ be a code.
- For each $i, j, t \in \{0, 1, \ldots, n\}$, define
$$\lambda_{i,j}^t = \left| \left\{ (X,Y,Z) \in \mathcal{C}^3 \;\middle|\; \begin{array}{l} d(X,Y) = i, d(X,Z) = j, \\ d(Y,Z) = i + j - 2t. \end{array} \right\} \right|$$
- Define $\binom{n}{a,b,c} = \frac{n!}{a!b!c!(n-a-b-c)!}$.
- If $\binom{n}{i-t,j-t,t} \neq 0$, let
$$x_{i,j}^t = \frac{1}{|\mathcal{C}| \binom{n}{i-t,j-t,t}} \lambda_{i,j}^t.$$
- If $\binom{n}{i-t,j-t,t} = 0$, let $x_{i,j}^t = 0$.

Schrijver SDP bound-Cont.

Remark:

- For each $i = 0, 1, \ldots, n$,

$$B_i = \binom{n}{i} x_{i,0}^0.$$

- Hence,

$$\sum_{i=0}^{n} \binom{n}{i} x_{i,0}^0 = |\mathcal{C}|.$$

Schrijver's result

Theorem (Schrijver, 2005) For $k = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$, the matrices

$$\left(\sum_{t=0}^{n} \beta_{i,j,k}^{t} x_{i,j}^{t} \right)_{i,j=k}^{n-k}$$

and

$$\left(\sum_{t=0}^{n} \beta_{i,j,k}^{t} (x_{i+j-2t,0}^{0} - x_{i,j}^{t}) \right)_{i,j=k}^{n-k}$$

are positive semidefinite, where

$$\beta_{i,j,k}^{t} := \sum_{u=0}^{n} (-1)^{u-t} \binom{u}{t} \binom{n-2k}{u-k} \binom{n-k-u}{i-u} \binom{n-k-u}{j-u}.$$

Schrijver's result-Cont.

Idea of the proof (1):

- Let $\mathcal{P}$ be the collection of all subsets of $\{1, 2, \ldots, n\}$ (which can be identified with $\mathcal{F}^n$).
- For i, j, t, let $M^t_{i,j}$ be the $\mathcal{P} \times \mathcal{P}$ matrix with

$$(M^t_{i,j})_{X,Y} = \begin{cases} 1 & \text{if } |X| = i, |Y| = j, |X \cap Y| = t \\ 0 & \text{otherwise} \end{cases} .$$

- Let

$$\mathcal{A}_n = \left\{ \sum_{i,j,t=0}^{n} c^t_{i,j} M^t_{i,j} \mid c^t_{i,j} \in \mathbb{C} \right\}.$$

Schrijver's result

Idea of the proof (2):

- $\mathcal{A}_n$ is called the Terwilliger algebra of the Hamming cube $\mathcal{P} \equiv \mathcal{F}^n$.
- There exists a unitary matrix U such that

$$U^*\mathcal{A}_nU = \left\{ \begin{pmatrix} C_0 & 0 & \cdots & 0 \\ 0 & C_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & C_m \end{pmatrix} \right\},$$

  where each C_k runs over all matrices of the form

$$\begin{pmatrix} B_k & 0 & \cdots & 0 \\ 0 & B_k & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & B_k \end{pmatrix}.$$

- Choosing 2 (suitable) positive semidefinite elements R, R' of $\mathcal{A}_n$, Schrijver get the desired result.

Schrijver semidefinite programming (SDP) bound

Schrijver SDP bound:

$$A(n,d) \leq \max \sum_{i=0}^{n} \binom{n}{i} x_{i,0}^{0}$$

subject to the matrices in the above Schrijver's Theorem are positive semidefinite for each $k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$ and subject to the following conditions on $x_{i,j}^{t}$.

- $x_{0,0}^{0} = 1.$
- $0 \leq x_{i,j}^{t} \leq x_{i,0}^{0}$ and $x_{i,0}^{0} + x_{j,0}^{0} \leq 1 + x_{i,j}^{t}$ for all $i, j, t \in \{0, \ldots, n\}$.
- $x_{i,j}^{t} = x_{i',j'}^{t'}$ if $(i', j', i' + j' - 2t')$ is a permutation of $(i, j, i + j - 2t)$.
- $x_{i,j}^{t} = 0$ if $\{i, j, i + j - 2t\} \cap \{1, \ldots, d - 1\} \neq \emptyset$.

II. Improvements on optimal codes

Some improvements on LP bound

Theorem (Mounits, Etzion, and Litsyn, 2002) Suppose $\mathcal{C}$ is a code of length n and minimum distance d. Let $\delta = d/2$. Then

- $B_i \leq A(n,d,i) \quad$ for $i = 1, \ldots, n$,

- $$B_{n-\delta} + \left\lfloor \frac{n}{\delta} \right\rfloor \sum_{i<\delta} B_{n-i} \leq \left\lfloor \frac{n}{\delta} \right\rfloor,$$

- $$B_{n-\delta-i} + [A(n,d,\delta+i) - A(n-\delta+i,d,\delta+i)]B_{n-\delta+i}$$
  $$+A(n,d,\delta+i)\sum_{j>i} B_{n-\delta+j} \leq A(n,d,\delta+i)$$

  for each i, $0 < i < \delta$.

Key point of improvements of LP bound

In Delsarte LP bound

- We deal with the distance distribution $\{B_i\}_{i=0}^{n}$.
- When counting pairs $(X, Y) \in \mathcal{C}^2$ such that $d(X, Y) = i$, we can first fix X and count Y (and then take sum over all X).
- This means we count the number of codewords Y at distance i from a fixed codeword X.
- This explains the appearance of $A(n, d, w)$'s in the improvements of LP bound.

Main result: Improved Schrijver SDP bound

In Schrijver SDP bound

- We deal with the "triple distance distribution" $\{x^t_{i,j}\}$.
- When counting triples $(X, Y, Z) \in \mathcal{C}^3$, we can first fix two codewords X, Y and then count Z.
- This means we count the number of codewords Z at fixed distances from X and Y.
- Hence, $A(n, \Lambda, w)$ would be involved, where Λ has two elements.
- Therefore, $T(w_1, n_1, w_2, n_2, d)$ would appear.

Main result: Improved Schrijver SDP bound-Cont.

Main Theorem: For each $i, j, t \in \{0, \ldots, n\}$ with $\binom{n}{i-t, j-t, t} \neq 0$,

$$x_{i,j}^t \leq \frac{T(t, i, j-t, n-i, d)}{\binom{i}{t}\binom{n-i}{j-t}} x_{i,0}^0.$$

Quick reason:

- We wish to count
$$\lambda_{i,j}^t = \left| \left\{ (X, Y, Z) \in \mathcal{C}^3 \;\middle|\; \begin{array}{l} d(X,Y) = i, d(X,Z) = j, \\ d(Y,Z) = i + j - 2t. \end{array} \right\} \right|.$$
- Double counting!! We first pick $(X, Y) \in \mathcal{C}^2$ such that $d(X, Y) = i$. Then the number of Z such that
$$(X, Y, Z) \in \left\{ (X, Y, Z) \in \mathcal{C}^3 \;\middle|\; \begin{array}{l} d(X,Y) = i, d(X,Z) = j, \\ d(Y,Z) = i + j - 2t. \end{array} \right\} \text{ is}$$

less that or equal to $A(n, \Lambda, d)$ where $\Lambda = \{X, j), (Y, i + j - 2t)\}$. And this value is $T(t, i, j - t, n - i, d)$.

- Now summing over all pairs $(X, Y) \in \mathcal{C}^2$ such that $d(X, Y) = i$.

Corollary: For each $j = 0, \ldots, n$,

$$x_{0,j}^0 \leq \frac{A(n, d, j)}{\binom{n}{j}}.$$

Main result: Improved Schrijver SDP bound-Cont.

Recall the conditions of Schrijver SDP bound

- $x^0_{0,0} = 1$.
- $0 \le x^t_{i,j} \le x^0_{i,0}$ and $x^0_{i,0} + x^0_{j,0} \le 1 + x^t_{i,j}$ for all $i, j, t \in \{0, \ldots, n\}$.
- $x^t_{i,j} = x^{t'}_{i',j'}$ if $(i', j', i' + j' - 2t')$ is a permutation of $(i, j, i + j - 2t)$.
- $x^t_{i,j} = 0$ if $\{i, j, i + j - 2t\} \cap \{1, \ldots, d - 1\} \neq \emptyset$.

Remark:

- The Theorem improves the condition $x^t_{i,j} \le x^0_{i,0}$ since $\frac{T(t,i,j-t,n-i,d)}{\binom{i}{t}\binom{n-i}{j-t}}$ is much less than 1 in general.
- The Corollary says that $x^0_{i,0} + x^0_{j,0} \le 1 + x^t_{i,j}$ for all i, j.

Main result: Improved Schrijver SDP bound-Cont.

More linear constraints

- Since $B_i = \binom{n}{i} x^0_{i,0}$ for all i, all linear constraints on B_i's (improvements of LP bound) can be used in SDP bound.
- More linear constraints on $x^t_{i,j}$'s have been (and are being) studied (but less hope to make more improvements).

New upper bounds on $A(n, d)$

Result

Improved upper bounds on $A(n,d)$

n	d	known lower bound	known upper bound	new upper bound	improved Schijver bound	Schrijver bound
18	8	64	72	71	71	80
19	8	128	135	131	131	142
20	8	256	256		262	274
25	8	4096	5421		5470	5477
26	8	4096	9275		9649	9697
27	8	8192	17099		17622	17768
27	10	512	1585		1764	1765
25	12	52	55		57	58
26	12	64	96		97	98

Remark

Final remark:

- We in fact get seven new upper bounds on $A(n,d)$ (for $n \leq 28$).
- However, five of them have been improved by D.C. Gijswijt, H.D. Mittelmann, A. Schrijver, "Semidefinite code bounds based on quadruple distances".
- Two new upper bounds are:

$$A(18,8) \leq 71 \quad \text{and} \quad A(19,8) \leq 131.$$

III. Improvements on optimal constant-weight codes

Progress on upper bounds on $A(n, d, w)$

In 1977: first tables for $n \leq 24$

- MacWilliams and Sloane: book "The Theory of Error-Correcting codes."

In 1978: updated tables

- Best, Brouwer, MacWilliams, Odlyzko, and Sloane: paper "Bounds for binary codes of length less than 25."

In 1987: more updated tables ($n \leq 27$)

- Honkala: his Licentiate thesis "Bounds for binary constant weight and covering codes."

Progress on bounds on $A(n, d, w)$

In 1990: update tables for $n \leq 28$

- Brouwer, Shearer, Sloane, and Smith: paper "A new table of constant weight codes."

In 2000: Improved upper bounds

- Agrell, Vardy, and Zeger: paper "Upper bounds for constant-weight codes."

In 2005: Using semidefinite programming

- Schrijver: paper "New code upper bounds from the Terwilliger algebra and semidefinite programming."

Our improvements

Main results We give two kind of improvements:

- One intersects the improvement of the Delsarte's linear programming bound in the paper "Upper bounds for constant-weight codes" (in 2000.)
- The other is an improvement of the Schrijver's semidefinite programming bound in the paper "New code upper bounds from the Terwilliger algebra and semidefinite programming" (in 2005.)

The first improvement

The key point

- In 2000, Agrell, Vardy, and Zeger showed that B_i and B_j (for suitable i and j) have a linear "relation". This gives a linear constraint on B_i and B_j which improves the Delsarte's LP bound.
- We show that B_i's for $i \in H$ (with $|H| \geq 2$) also have a linear "relation". For $n \leq 28$, with $|H| = 3$, new upper bounds on $A(n, d, w)$ are obtained.
- The improvement comes from the observation that: the existence of a codeword at distance i from a fixed codeword X will **"effect"** not only the number of codewords at distance j from X (showed by Agrell, Vardy, and Zeger) but also the number of codewords at distance k from X, etc..

The first improvement-Cont.

Example

- Consider $A(27, 8, 13)$.
- By the result of Agrell, Vardy, and Zeger,

$$B_{22} + 6B_{24} \leq 26, \quad B_{22} + 26B_{26} \leq 26, \quad B_{24} + B_{26} \leq 1.$$

- Our result gives

$$B_{22} + 6B_{24} + 26B_{26} \leq 26, \quad B_{24} + B_{26} \leq 1.$$

- We get $A(27, 8, 13) \leq 11904$. This improves the upper bound of Agrell, Vardy, and Zeger: $A(27, 8, 13) \leq 11991$, and the best upper bound of Schrijver: $A(27, 8, 13) \leq 11981$.

The second improvement

Schrijver's semidefinite programming (SDP) bound

- Schrijver's (SDP) bound is based on the "triple distance distribution" of constant-weight codes.

The second improvement

"Triple distance distribution" of constant-weight codes

- Let $\mathcal{C}$ be an (n, d, w) constant-weight code. Let $v = n - w$. For each t, s, i, j, define

$$y_{i,j}^{t,s} = \frac{1}{|\mathcal{C}| \binom{w}{i-t,\, j-t,\, t} \binom{v}{i-s,\, j-s,\, s}} \mu_{i,j}^{t,s},$$

  where $\mu_{i,j}^{t,s}$ is the number of triples $(X, Y, Z) \in \mathcal{C}^3$ with $d(X, Y) = 2i$, $d(X, Z) = 2j$, $d(Y, Z) = 2(i + j - t - s)$, and $d(X + Y, Z) = w + 2t - 2s$.

- Set $y_{i,j}^{t,s} = 0$ if either $\binom{w}{i-t,\, j-t,\, t} = 0$ or $\binom{v}{i-s,\, j-s,\, s} = 0$.

The second improvement-Cont.

Our improvement

- Schrijver showed that $y_{i,j}^{t,s} \leq y_{i,0}^{0,0}$ and $y_{i,0}^{0,0} + y_{j,0}^{0,0} \leq 1 + y_{i,j}^{t,s}$ for every t, s, i, j.
- We improve these by showing that

$$y_{i,j}^{t,s} \leq \frac{T(t, i, j-t, w-is, i, j-s, v-i, d)}{\binom{i}{t}\binom{w-i}{j-t}\binom{i}{s}\binom{v-i}{j-s}} y_{i,0}^{0,0}$$

for every t, s, i, j, where $T(w_1, n_1, w_2, n_2, w_3, n_3, w_4, n_4, d)$ can be defined similarly as $A(n, d)$ (the difference is that each codeword must have the form

$$X = (X_1, X_2, X_3, X_4) \in \mathcal{F}^{n_1} \times \mathcal{F}^{n_2} \times \mathcal{F}^{n_3} \times \mathcal{F}^{n_4}$$

with $wt(X_i) = w_i$ for $i = 1, \ldots, 4$.

The second improvement-Cont.

The key point

- When counting $(X, Y, Z) \in \mathcal{C}^3$, we fix (X, Y) first and then count Z. Then we can see that the improvement (naturally) comes from the definition of $A(n, \Lambda, d)$ for a "special" case of $|\Lambda| = 4$.

New upper bounds on $A(n,d,w), d = 6$

Tables of new upper bounds on $A(n,d,w)$ For $n \leq 28$, there are 21 new upper bounds on $A(n,d,w)$ which are listed below.

Table 1: New upper bounds for $A(n,d,w)$

n	d	w	lower bound	upper bound	new upper bound	Schrijver bound
20	6	8	588	1107	1106	1136

New upper bounds on $A(n, d, w), d = 8$

n	d	w	lower bound	upper bound	new upper bound	Schrijver bound
22	8	10	616	634	630	634
23	8	9	400	707	703	707
26	8	11	1988	5225	5208	5225
27	8	9	1023	2914	2911	2918
27	8	11	2404	7833	7754	7833
27	8	12	3335	10547	10472	10697
27	8	13	4094	11981	11904	11981
28	8	11	3773	11939	11896	12025
28	8	12	4927	17011	17010	17011
28	8	13	6848	21152	21148	21152

New upper bounds on $A(n, d, w), d = 10$

n	d	w	lower bound	upper bound	new upper bound	Schrijver bound
23	10	9	45	81	79	82
25	10	11	125	380	379	380
25	10	12	137	434	433	434
26	10	11	168	566	565	566
26	10	12	208	702	691	702
27	10	11	243	882	871	882
27	10	12	351	1201	1191	1201
27	10	13	405	1419	1406	1419
28	10	11	308	1356	1351	1356

Table 2: New upper bounds for $A(n, d, w), d = 12$

n	d	w	lower bound	upper bound	new upper bound	Schrijver bound
25	12	10	28	37	36	37

Thank you!